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Strict weak orderings in semiotics

1. A strict weak ordering is a binary relation < on a set S that is a strict partial order, i.e. a transitive relation that is irreflexive, or equivalently, that is asymmetric, in which the relation "neither a < b nor b < a" is transitive. The equivalence classes of this "incomparability relation" partition the elements of S, and are totally ordered by <. Conversely, any total order on a partition of S gives rise to a strict weak ordering in which x < y if and only if there exists sets A and B in the partition with x in A, y in B, and A < B in the total order (Roberts 1979).

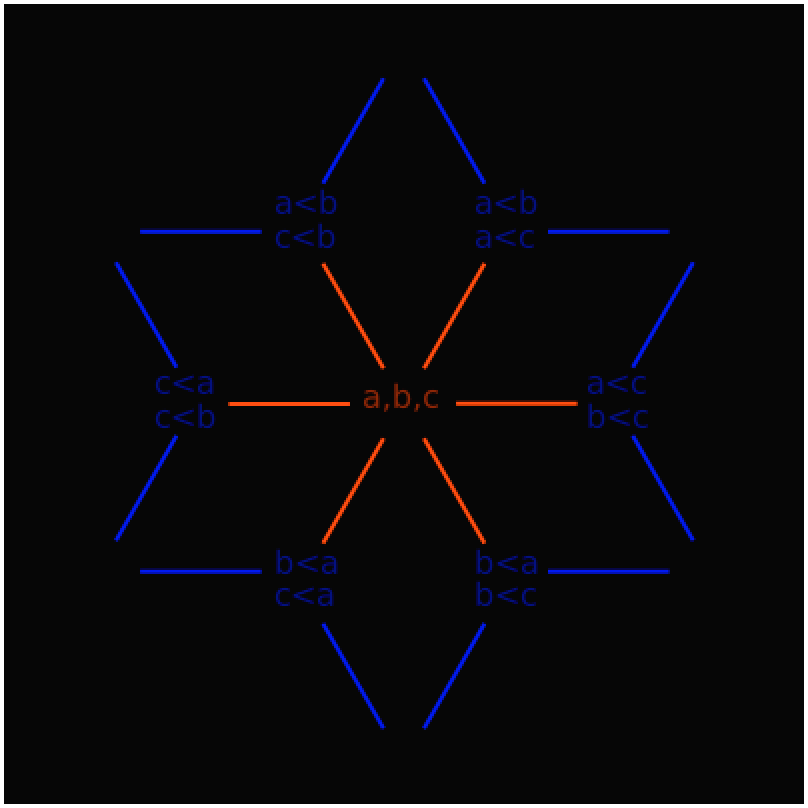
A strict weak ordering has the following properties. For all x and y in S,

- For all x, it is not the case that x < x (irreflexivity).
- For all x ≠ y, if x < y then it is not the case that y < x (asymmetry).
- For all x, y, and z, if x < y and y < z then x < z (transitivity).
- For all x, y, and z, if x is incomparable with y, and y is incomparable with z, then x is incomparable with z (transitivity of equivalence) ≡ If x < y, then for all z either x < z or z < y or both

2. As a first example, we show the 13 possible strict weak orders on the set $SR_{3,3}$ = {.1., .2., .3.}, or simplified {1, 2, 3}, of the triadic-trichotomic sign relation:

{{{1}, {2}, {3}}, {{1}, {3}, {2}}, {{2}, {1}, {3}}, {{3}, {1}, {2}}, {{2}, {3}, {1}}, {{3}, {2}, {1}}, {{1}, {2, 3}}, {{2}, {1, 3}}, {{3}, {1, 2}}, {{1, 2}, {3}}, {{1, 3}, {2}}, {{2, 3}, {1}}, {{1, 2, 3}}}

that can be displayed with the following graph:



http://en.wikipedia.org/wiki/Strict_weak_ordering

3. Strict weak orders are very closely related to total preorders or (non-strict) weak orders, and the same mathematical concepts can be modeled equally well with total preorders. A total preorder or weak order is a preorder that is total; that is, no pair of items is incomparable. A total preorder $\leq$ satisfies the following properties:

- For all x, y, and z, if x $\lesssim$ y, and y $\lesssim$ z, then x $\lesssim$ z (transitivity).
- For all x and y, x $\lesssim$ y or y $\lesssim$ x (totality).
- Hence: For all x, x $\lesssim$ x (reflexivity).

A total order is a total preorder which is antisymmetric, in other words, which is also a partial order (Roberts 1979). The number of total preorders is given by the Fubini numbers or ordered Bell numbers:

n	all	trans.	refl.	preor.	part. order	total preord.	total order	equiv. rel.
0	1	1--	1	1	1	1	1	1
1	2	2--	1	1	1	1	1	1
2	16	13	4	4	3	3	2	2
3	512	171	64	29	19	13	6	5
4	65‘536	3‘994	4‘096	355	219	75	24	15

We have already shown the 13 total preorders for the set {.1., .2., .3.} with n = 3. For n = 4, for which we can take as examples $SR_{4,3}$ = {0., .1., .2., .3.} or $SR_{4,4}$ = {.0., .1., .2., .3.} (cf. Toth 2008a), we have 75 total preorders:

- 1 partition of 4 sets, giving 1 total preorder, i.e. each element is related to each element: {0, 1, 2, 3}
- 7 partitions of 2 sets, giving 14 total preorders: {{0, 3}, {1, 2}}, {{1, 2}, {0, 3}}, {{0}, {1, 2, 3}}, {{1, 2, 3}, {0}}, {{0, 1, 3}, {2}}, {{2}, {0, 1, 3}}, {{0, 2}, {1, 3}}, {{ 1, 3}, {0, 2)}, {{0, 1, 2}, {3}}, {{3}, {0, 1, 2}}, {{0, 2, 3}, {1}}, {{1}, {0, 2, 3}}, {{0, 1}, {2, 3}}, {{2, 3}, {0, 1}}
- 6 partitions of 3 sets, giving 36 total preorders:
  {{0}, {1, 2}, {3}}, {{0}, {3}, {1, 2}}, {{1, 2}, {0}, {3}}, {{1, 2}, {3}, {0}},
  {{3}, {1, 2}, {0}}, {{3}, {0}, {1, 2}}
  {{0, 3}, {1}, {2}}, ...
  {{0}, {1, 3}, 2}}, ...
  {{0, 2}, {1}, {3}}, ...
  {{0, 1}, {2}, {3}}, ...
  {{0}, {1}, {2, 3}}, ...
- 1 partition of 1 set, giving 24 total preorders, i.e. the total orders:
  {{1}, {2}, {3}, {4}} and all permutations

The number of ordered partitions T_n of {1, 2, ..., n} is calculated recursively by

$$T_n = \sum_{i=0}^{n-1} \binom{n}{i-} T_i.$$

A strict weak order that is trichotomous is called a strict total order, i.e. exactly one of the relations a < b, b < a, a = b holds. E.g., for the set of the triadic-trichotomic sign classes based on $SR_{3,3}$, = (3.a 2.b 1.c) with a ≤ b ≤ c, we get the following sets of pairs of dyads:

(a < b): {(3.1, 2.2), (3.1, 2.3), (3.2, 2.3), (2.1, 1.2), (2.1, 1.3), (2.2, 1.3)}

(a = b): {(3.1, 2.1), (3.2, 2.2), (3.3, 2.3), (2.1, 1.1), (2.2, 1.2), (2.3, 1.3)}

However, the relation (b < a) does not hold in $SR_{3,3}$ as long a the trichotomic semiotic inclusion order is valid; therefore, we find this type of order only in the 17 complementary sign classes out of the total amount of 27 triadic-trichotomic sign classes (cf. Toth 2008b)

(b < a): {(3.2 2.1), (3.3, 2.1), (3.3, 2.2), (2.2, 1.1), (2.3, 1.1), (2.3, 1.2)}.

Moreover, this order type is present as main diagonal in the semiotic matrix over $SR_{3,3}$:

1.1	1.2	1.3
2.1	**2.2**	2.3
3.1	3.2	**3.3**

This so-called Genuine Category Class (cf. Bense 1992, pp. 27 ss.) (3.3 2.2 1.1) has trichotomic order (3.a 2.b 1.c) with $a > b > c$ which is at the same time trichotomous. In the set of the 10 sign classes, it shares trichotousness only with the sub-set of the homogeneous sign classes on the one side {(3.1 2.1 1.1), (3.2 2.2 1.2), (3.3 2.3 1.3) with trichotomic order $a = b = c$)} and with the eigen-real sign class (3.1 2.2 1.3) with trichotomic order ($a < b < c$) on the other side; the other 6 sign classes are of mixed trichotomic order and thus not trichotomous.

Bibliography

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